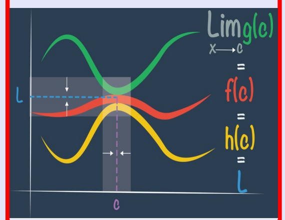
Math 261

Fall 2022

Lecture 33



Verify all conditions by Rolle's theorem, and find all numbers c that satisfy the conclusion of Rolle's Thrm for

$f(x) = \sqrt{x} - \frac{1}{3}x$ on $[0,9]$

✓ 1) $f(x)$ cont. $[a,b]$

✓ 2) $f(x)$ diff. (a,b)

✓ 3) $f(a) = f(b)$

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$ on $(0,9)$

$f(0) = 0$

$f(9) = \sqrt{9} - \frac{1}{3}(9) = 0$

Concl. $f'(c) = 0$

$\frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$ $\qquad \frac{1}{2\sqrt{c}} = \frac{1}{3}$ $\qquad 2\sqrt{c} = 3$

$\sqrt{c} = \frac{3}{2}$

$c = \frac{9}{4} = 2.25$

$f(x) = \frac{1}{x}$, $[1,3]$

Show that $f(x)$ satisfy all conditions by MVT, then find all numbers c that is in the conclusion of MVT.

√1) $f(x)$ cont. $[a,b]$

√2) $f(x)$ diff. (a,b)

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$\frac{-1}{c^2} = \frac{\frac{1}{3} - 1}{3 - 1}$

$f(x) = \frac{1}{x}$ $[1,3]$

$f'(x) = \frac{-1}{x^2}$ $(1,3)$

$f(1) = 1$, $f(3) = \frac{1}{3}$

$\frac{-1}{c^2} = \frac{-\frac{2}{3}}{2}$

$\frac{2}{3}c^2 = 2$

$c^2 = 3$

$c = \pm\sqrt{3}$

$c = \sqrt{3}$ is in $(1,3)$

Suppose $f(1) = 10$, $f'(x) \geq 2$ for $1 \leq x \leq 4$,

$\boxed{\text{how small can } f(4) \text{ be}}$?

If $f(x)$ is diff., then $f(x)$ is cont.

$f'(x) \geq 2$ on $[1,4]$ → $f'(x)$ is diff. & $f(x)$ is cont.

we can use MVT

$f'(c) = \frac{f(b) - f(a)}{b - a}$

for our problem $[1,4]$

$\frac{f(4) - f(1)}{4 - 1} = f'(c)$

$\frac{f(4) - 10}{3} = f'(c) \geq 2$

$\frac{f(4) - 10}{3} \geq 2$

$f(4) - 10 \geq 6$

$\boxed{f(4) \geq 16}$

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find a function $f(x)$ if exists such that $f(0)=-1$, $f(2)=4$, and $f'(x)\le 2$ for all x.

$f'(x) \le 2 \quad \forall x$ $\quad$ $f(x)$ is diff., then $f(x)$ is cont. for $\forall x$.

By MVT
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Our interval is $[0,2]$

$$\frac{f(2)-f(0)}{2-0} = f'(c)$$

$$\frac{4-(-1)}{2} = f'(c) \le 2$$

$$\frac{5}{2} \le 2$$

$$2.5 \le 2$$

false

No such function on $[0,2]$ such that $f(0)=-1$, $f(2)=4$, and $f'(x)\le 2$

Suppose $3 \le f'(x) \le 5$ for all values of x,

Show that $\quad 18 \le f(8) - f(2) \le 30.$

use $[2,8]$

$f(x)$ is cont.
$f(x)$ is diff. $\Rightarrow$ $f(x)$ is diff. $\Rightarrow$ cont.

By MVT $\quad f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\frac{f(8)-f(2)}{8-2} = f'(c)$$

$$\frac{f(8)-f(2)}{6} = f'(c)$$

but $\quad 3 \le f'(x) \le 5 \rightarrow 3 \le \dfrac{f(8)-f(2)}{6} \le 5$

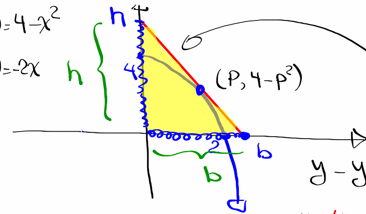
multiply by 6

$$18 \le f(8) - f(2) \le 30$$

What is the $\boxed{\text{Smallest possible area}}$ of the triangle is cut off by QI whose hypotenuse is tangent to the parabola $y = 4 - x^2$?

$f(x) = 4 - x^2$

$f'(x) = -2x$



$A = \dfrac{x\text{-int} \cdot y\text{-int}}{2}$

$m = f'(p)$

$m = -2p$

$y - y_1 = m(x - x_1)$

$y - (4 - p^2) = -2p(x - p)$

$y - 4 + p^2 = -2px + 2p^2$

$\boxed{y = -2px + p^2 + 4}$

x-Int.

$y = 0$

$-2px + p^2 + 4 = 0$

$p^2 + 4 = 2px$

$x = \dfrac{p^2+4}{2p}$ ← x of x-Int.

Y-Int

$x = 0$

$y = -2p(0) + p^2 + 4$ $\quad y = p^2 + 4$ ← Y of Y-Int.

$$\text{Area} = \frac{\frac{p^2+4}{2p} \cdot (p^2+4)}{2} = \frac{(p^2+4)(p^2+4)}{2p \cdot 2} = \frac{p^4 + 8p^2 + 16}{4p}$$

$$A(p) = \frac{p^4}{4p} + \frac{8p^2}{4p} + \frac{16}{4p}$$

$$A(p) = \frac{p^3}{4} + 2p + \frac{4}{p}$$

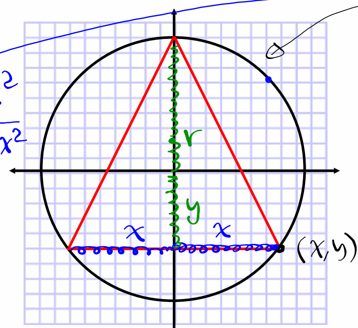
optimize this to find smallest area.

Find the $\boxed{\text{dimensions of the isosceles}}$ triangle of largest area that can be inscribed in a Circle with radius r.

$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

$y = \sqrt{r^2 - x^2}$



base $= 2x$

height $= y + r$

$\text{Area} = \dfrac{bh}{2}$

$= \dfrac{2x \cdot (y + r)}{2}$

$\text{Area} = x(y + r)$

$A(x) = x\left(\sqrt{r^2 - x^2} + r\right)$

$\dfrac{d}{dx}\left(\sqrt{r^2 - x^2}\right) =$

$\dfrac{d}{dx}\left[(r^2 - x^2)^{1/2}\right]$

$\dfrac{1}{2} \cdot (r^2 - x^2)^{-1/2} \cdot -2x$

$A'(x) = 1\left(\sqrt{r^2 - x^2} + r\right) + x\left(\dfrac{-x}{\sqrt{r^2 - x^2}}\right)$

$A'(x) = \sqrt{r^2 - x^2} + r - \dfrac{x^2}{\sqrt{r^2 - x^2}}$

$A'(x) = \dfrac{r^2 - x^2 + r\sqrt{r^2 - x^2} - x^2}{\sqrt{r^2 - x^2}} = \dfrac{r^2 - 2x^2 + r\sqrt{r^2 - x^2}}{\sqrt{r^2 - x^2}}$

$A'(x) = 0$

$r^2 - 2x^2 + r\sqrt{r^2 - x^2} = 0$

$r\sqrt{r^2 - x^2} = 2x^2 - r^2$ Square both Sides

$r^2(r^2 - x^2) = 4x^4 - 4x^2r^2 + r^4$

$\cancel{r^4} - r^2x^2 = 4x^4 - 4x^2r^2 \cancel{+r^4}$

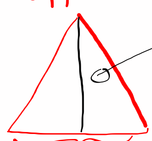
$4x^4 - 3x^2r^2 = 0$

$x^2(4x^2 - 3r^2) = 0$

$x = 0$ $\qquad$ $x = \frac{r\sqrt{3}}{2}$

Not applicable

we can verify $f'(x)$ $+$ $\frac{r\sqrt{3}}{2}$ $-$



y+r

$h = y + r$

$\boxed{h = \frac{3}{2}r}$

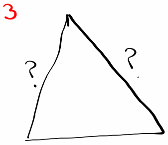
$2x = r\sqrt{3}$

$x^2 + y^2 = r^2$

$\left(\frac{r\sqrt{3}}{2}\right)^2 + y^2 = r^2$

$\frac{3r^2}{4} + y^2 = r^2$

$y^2 = \frac{r^2}{4}$



Find each Side?

$y = \frac{r}{2}$